

APPENDIX E

SHORE-TO-PIER ACCESS RAMP PRELIMINARY DESIGN CALCULATIONS

This appendix contains the following calculations.

- Verification Study – Torsional Flexibility of the Access Ramps

Verified that the preliminary access ramp structure is sufficiently flexible in torsion to accommodate the roll displacement of the MHP within the allowable stress limits.

- Hand Calculation for Stability of the Upper Chord of a Low-Truss Bridge

Stability of the upper chord is not a problem according to the calculation.

VERIFICATION STUDY



Verification Study - Torsional Flexibility of the Access Ramps

Assumption

Member sizes: (Determined by Phil B.)

Chord (top and bottom)	– TS 8x8x1/2
Verticals	- TS 8x6x3/8
Diagonals	- TS 8x4x3/8
Floor Beams	- W24x68

Crane Load

Using TM9100 crane (100 Ton crane) ***

Front Axle – 49 kips

Back Axle – 77 kips

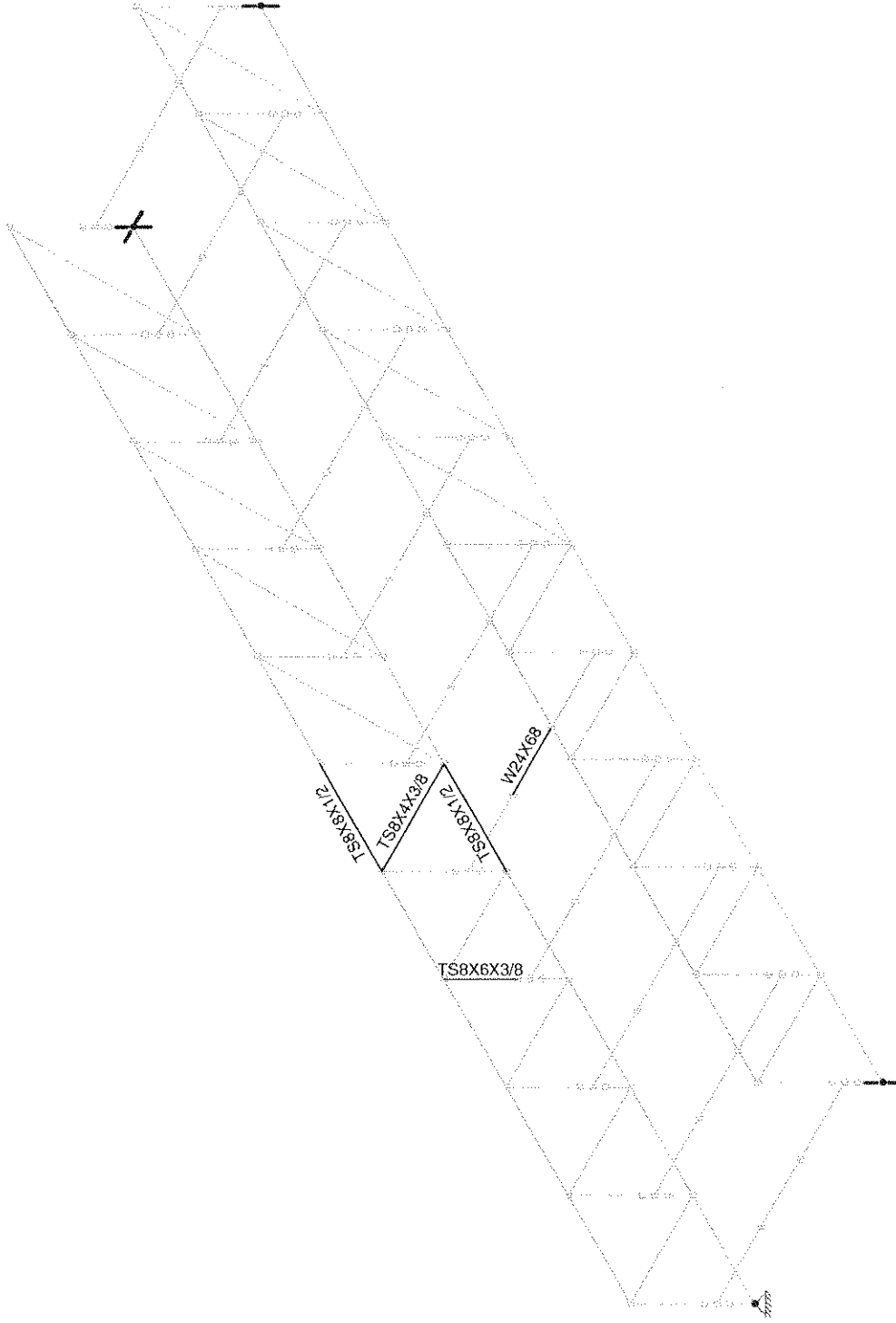
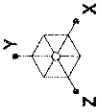
Modeling

- Used RISA 3-D
- All connections are Moment Connection
- Support Condition:
 - MHP side – One support - Pinned condition
 - The other support – Only Vertical
 - Yard Side - One Support – Only Vertical
 - The other – Vertical and Lateral (Transverse)
- Crane Load location
 - @ Mid Span
 - @ Near End (Back Axle to be on 1st Interior Span)
- Support Displacement
 - To represent the MHP Rolling, Displace One MHP support for 8" (Equivalent of 2 degree Roll)

*** In final design the ramp analysis will include analysis for 140T Crane Loads of MIL-HDBK 1025/1

Preliminary Size to start Later Analysis

Chord (top and bottom)	– TS 8x8x5/8 or TS 10x10x1/2
Verticals	- TS 8x6x3/8 or TS10x5x1/2
Diagonals	- TS 8x4x3/8 or TX10x5x3/8
Floor Beams	- W24x104 or W24x117



Louis: BLC 4, LL-CRANE2
Solution: LC 4, DL+LL2+8"D

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Liz Davey Kayoko Price

A01047

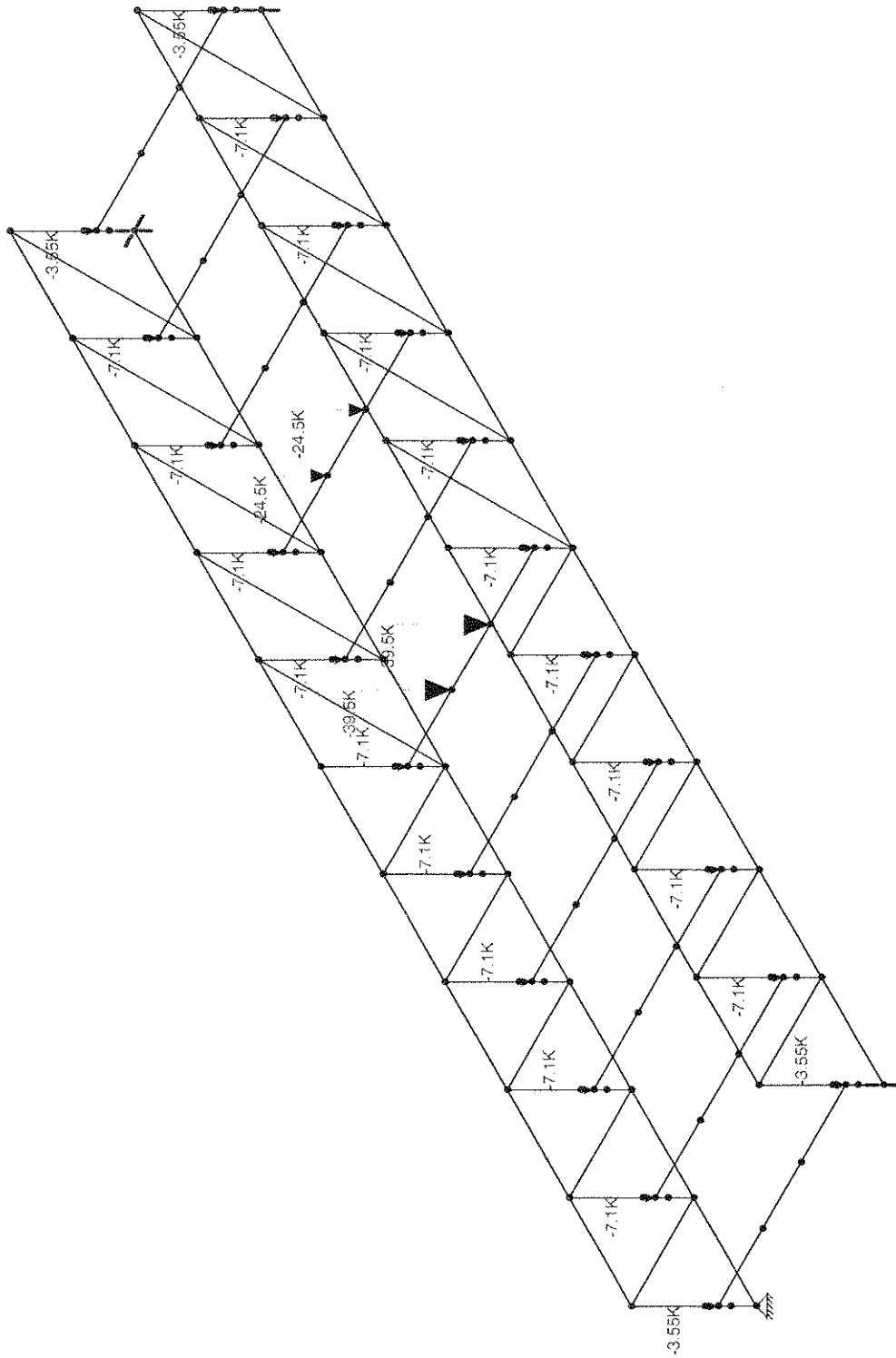
Torsionally Flexible Bridge

TRUSS TRIAL MEMBERS

June 25, 2001

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Loads: LC 5, DL+LL3+8"D
Solution: LC 5, DL+LL3+8"D

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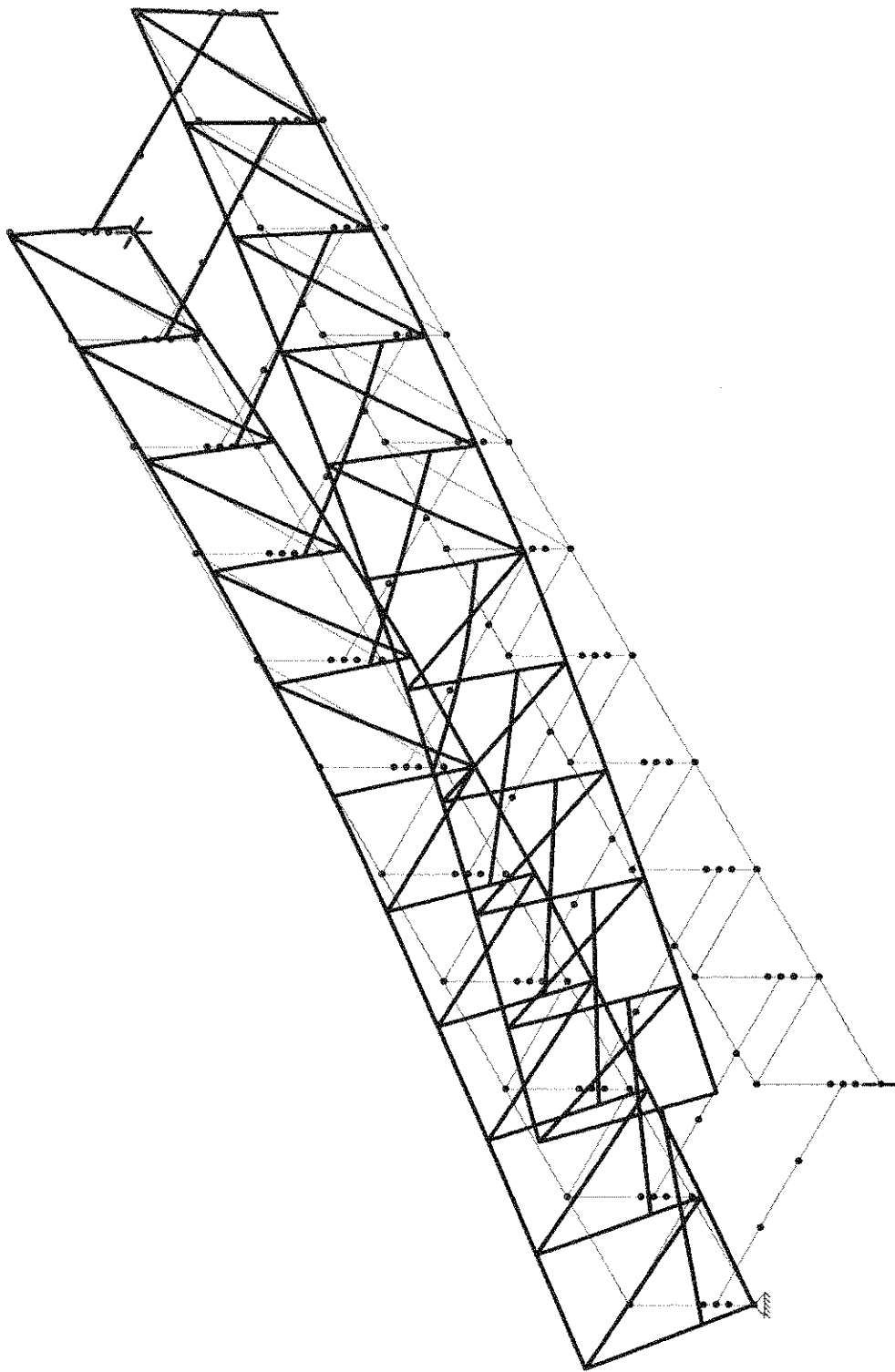
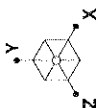
Torsionally Flexible Bridge

DL+8" TWIST+100 Ton Crane @ Mid Span [LOADS]

June 6, 2001

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Solution: LC 5, DL+LL3+8'D

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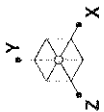
Torsionally Flexible Bridge

DL+8"TWIST+100 Ton Crane @ Mid Span [Displacement Plot]

June 6, 2001

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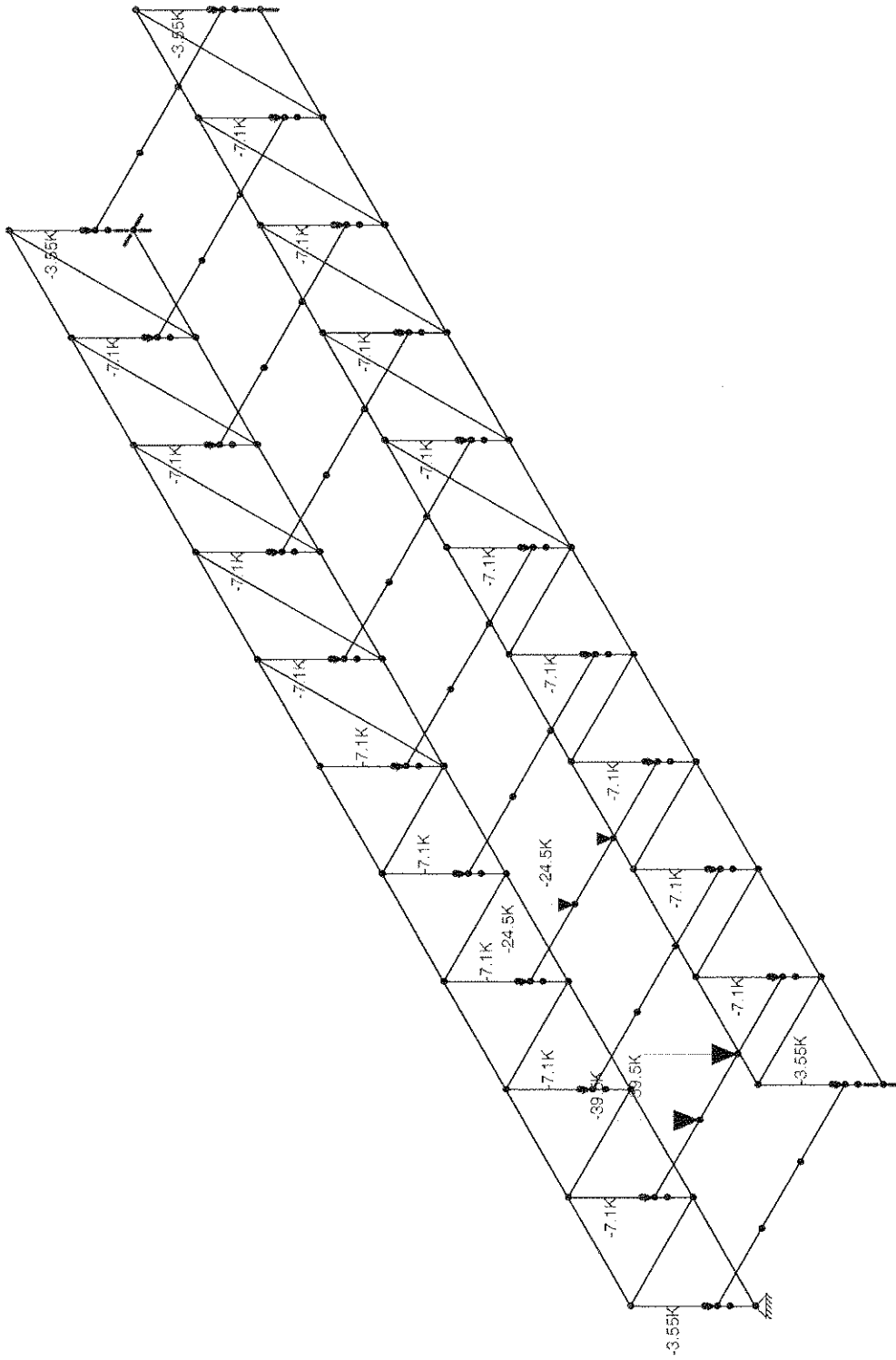
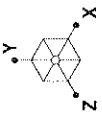
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DL+8" TWIST +100T Crane @ Midspan [CODE CHECK]



Loads: LC 4, DL+LL2+8'D
Solution: LC 4, DL+LL2+8'D

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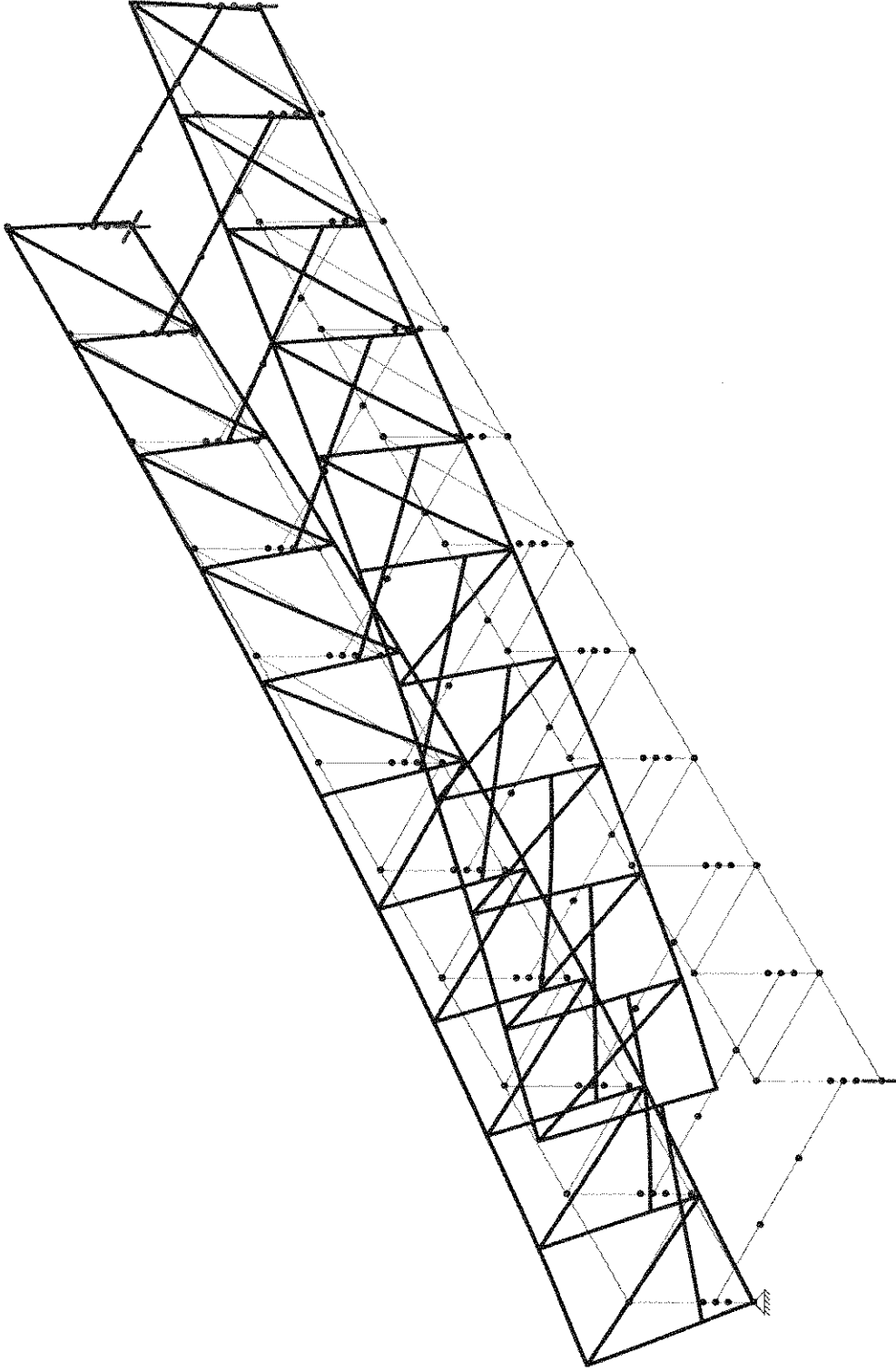
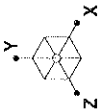
Torsionally Flexible Bridge

DL+8"TWIST+100Ton Crane @ Near End [LOADS]

June 6, 2001

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Solution: LC 4, DL+LL2+8"D

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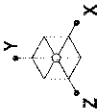
Torsionally Flexible Bridge

DL+8"TWIST+100Ton Crane @ Near End [Displacement Plot]

June 6, 2001

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FLEXT_BR1r3d.r3d



DL+8"TWIST+100T Crane @ Near End [CODE CHECK]



Project	MHP PH II	Sheet	_____ of _____
		Job Number	A01047
Subject	RAMP DESIGN	Designer	KAP
		Date	5/9/01 updated

[QUICK COMPARISON WITH 140T CRANE AND 100T (TM9100 CRANE)

Preliminary member size of the ramp bridge structure were based on the TM9100 Crane (100T). Member sizes have to be bumped up for the heavier loaded crane, 140T crane.

CRANE:

	Front Axle Wt.	Rear Axle Wt	Total WT
140T	26,844 lbs	141,976 lbs	168,820 lbs
TM9100	49,000 lbs	77,000 lbs	126,000 lbs
Factor	0.55	1.84	1.34

(140T/TM9100)

Result of Quick Re-calculation of Loads in the member:

	Member Force	TM9100	140T	Factor
T & B Chord	Max Axial Load (P)	244 k	295 k	1.21
Diag and Vertical	Max Shear (V)	92 k	114 k	1.24
Floorbeam	Max Moment (M)	291.5 k-ft	512 k-ft	1.76

Note: Factors for the loading is little less than the Crane Loads, it is because the member Loads are calculated including the approximate DL of the bridge.

Preliminary Member Size for 140 T Crane (For Later Analysis):

	TM9100	140T
T & B Chord	TS 8x8x1/2	TS 8x8x5/8 or TS 10x10x1/2
Vertical	TS 8x6x3/8	TS 8x6x3/8 or TS10x5x1/2
Diagonal	TS 8x4x3/8	TS 8x4x3/8 or TX10x5x3/8
Floorbeam	W24x64	W24x104 or W24x117

STABILITY OF UPPER CHORD



Project	MHP PH II	Sheet	_____ of _____
		Job Number	A01047
Subject	Trosionally Flexible Bridge	Designer	KAP
		Date	6/6/01 updated

STABILITY OF THE UPPER CHORD OF A LOW-TRUSS BRIDGE

(Per Theory of Elastic Stability by S. Timoshenko)

ASSUMPTION

" If the propostion of the compressed chord and verticals of the bridge are such that the half-wave length of the buckled chord is large in comparison with one panel length of the bridge (say the half-wave length is not less than three panels) a great simplification of the problem can be obtained by replacing the elastic supports by an equivalent elastic foundation"

Vertical Member	H =	9.8 ft
Panel Width	W =	10 ft
Total Bridge Length	L =	98 ft

Therefore, Half-wave $L = 98'/2 = 49$ ft less than $3 \times 10' = 30$ ft ok

Determine Modulus of the equivalenet elastic foundtion

$$B = \frac{R_o}{C}$$

$$R_o = \frac{1}{\frac{a^3}{3EI_1} + \frac{(a+b)^2 d}{2EI_2}}$$

a =	9.8 ft - 3' ft (to cL of BM) - 1' (1/2 BM) =	5.8ft
b =	1.0 ft (1/2 BM)	
c =	10 ft	
d =	20 ft	

$I_1 =$	I for Vertical, TS 8x6x3/8 => $I_x = 83.7 \text{ in}^4$
$I_2 =$	I for Floor BM, W24x68 => $I_x = 1830 \text{ in}^4$

$$R_o = \frac{1}{\frac{(5.8' \times 12)^3}{(3)(29000 \text{ ksi})(83.7 \text{ in}^4)} + \frac{(6.8' \times 12)^2 (20' \times 12)}{(2)(29000 \text{ ksi})(1830 \text{ in}^4)}} = 16.3 \text{ kip/in}$$

$$B = \frac{16.3 \text{ kip/in}}{(10' \times 12) \text{ in}} = \underline{\underline{0.136 \text{ ki/in}^2}}$$



Project MHP PH II Sheet _____ of _____
 Job Number A01047
 Subject Torsionally Flexible Bridge Designer KAP
 Date 6/6/01 updated

Determine Reduced Length, L

Use Table 9 of Ref - Table for Calculating the reduced length, L

$$\frac{B I^4}{16 EI}$$

$$\begin{aligned} B &= 0.136 \text{ kip/in}^2 \\ I &= 9.8' \times 12 = 1176 \text{ in} \\ I_{\text{chord}} &= 131 \text{ in}^4 \text{ (TS 8x8x1/2)} \end{aligned}$$

$$\text{Therefore } \frac{B I^4}{16 EI} = \frac{(0.136 \text{ kip/in}^2) (1176 \text{ in})^4}{16 (29000 \text{ ksi})(131 \text{ in}^4)} = 4279$$

Table 9 goes up to 1000. Let's use $L/I = 0.174$ Conservative!!

$$L \approx (98')(0.174) = 17\text{ft}$$

Determine Critical Load

$$\frac{q_o L}{4} \text{ critical} = \frac{(\pi^2 EI)}{L^2}$$

$$= \frac{\pi^2 (29000 \text{ ksi}) (131 \text{ in}^4)}{(17 \text{ ft} \times 12)^2}$$

$$= 901 \text{ kips} \quad \text{Less than P analysis} = 230 \text{ kips}$$

ok

H.W. BIRKELAND

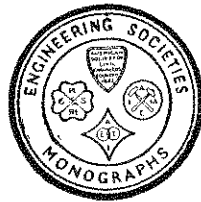
1943

THEORY OF ELASTIC STABILITY

BY

S. TIMOSHENKO

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NEW YORK AND LONDON

1936

$$(ql)_{cr} = \frac{7.84EI}{l^2}$$

This value practically coincides with that given in (83).

By using the energy method we can consider also a vertical bar hinged at the ends and submitted to the action of its own weight ql in addition to compressive forces P applied at the ends (Fig. 77). The critical values of P can be represented by the equation

$$P_{cr} = \frac{mEI}{l^2}, \quad (85)$$

in which the numerical factor m depends on the value of the ratio

$$n = ql \div \frac{\pi^2 EI}{l^2}$$

Several values of the factor m are given in Table 8.

TABLE 8.—VALUES OF m IN EQ. (85)

n	0	0.25	0.50	0.75	1.0	2.0	3.0
m	π^2	8.63	7.36	6.08	4.77	-.657	-4.94

It is seen, from this table, that a satisfactory approximation for the critical load P is obtained by assuming that one-half of the weight ql of the bar is applied at the top, *i.e.*, by taking

$$P_{cr} = \frac{\pi^2 EI}{l^2} - \frac{ql}{2}$$



FIG. 77.

When n is 2 or larger, P_{cr} is negative, which indicates that in such cases tensile forces P should be applied at the ends to prevent the bar from lateral buckling.

The energy method can be applied advantageously in various cases of distributed compressive loads acting on a bar. In this way the integration of equations with variable coefficients, requiring the use of infinite series, is replaced by the simple problem of finding the minimum of a certain expression, such as the right side of Eq. (7) above. By increasing the number of terms in the expression for the deflection curve, as in Eq. (2) above, the accuracy of the solution can be increased, although the first approximation is usually sufficiently accurate for practical applications. Later, we shall apply this method to a discussion of the stability of the upper chord of a low-truss bridge.

24. Stability of the Upper Chord of a Low-truss Bridge.—In a low-truss bridge (sometimes called a pony truss), there is no bracing in the upper horizontal plane (Fig. 78) and the upper chord is in the condition of a compressed bar, the lateral buckling of which is resisted by the elastic reactions of the vertical and diagonal members. At the supports there are usually frames of considerable rigidity so that the ends of the chord may be con-

sidered as immovable in a lateral direction. Thus the upper chord may be considered as a bar with hinged ends compressed by forces distributed along its length and elastically supported at intermediate points. A general method of solving problems of this kind is discussed in Art. 20.

However, the amount of work necessary to obtain the critical value of the compressive force increases rapidly with the number of elastic supports.¹ The stability of the compressed chord can be increased by increasing the rigidity of the lateral supports. For a constant cross section of the chord and a constant compressive force, the minimum rigidity, at which the

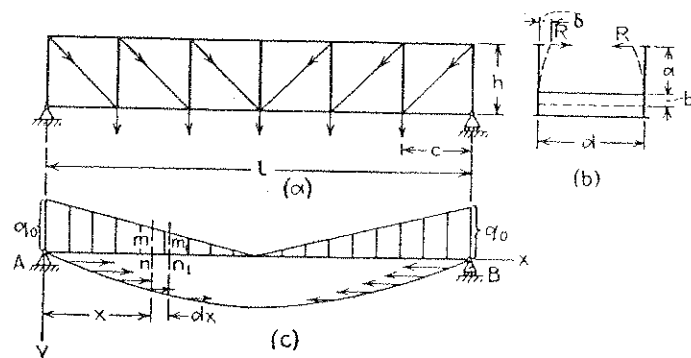


FIG. 78.

supports begin to behave as though they were absolutely rigid, is found from Eq. (72) of Art. 20 (see p. 107). If the proportions of the compressed chord and verticals of the bridge (Fig. 78) are such that the half-wave length of the buckled chord is large in comparison with one panel length of the bridge (say the half-wave length is not less than three panels), a great simplification of the problem can be obtained by replacing the elastic supports by an equivalent elastic foundation (see Art. 20) and replacing the concentrated compressive forces, applied at the joints, by a continuously distributed load. Assuming that the bridge is uniformly loaded, the compressive forces transmitted to the chord by the diagonals are proportional to the distances from the middle of the span, and the equivalent compressive load distribution is as shown in Fig. 78c by the shaded areas.

In calculating the modulus β of the elastic foundation, equivalent to the elastic resistance of the verticals,² it is necessary to establish the relation between the force R , applied at the top of a vertical (Fig. 78b) and the deflection that would be produced if the upper chord were removed. If

¹ Several numerical examples of calculations of the stability of a compressed chord as a bar on elastic supports can be found in the book by H. Müller-Breslau, "Graphische Statik," vol. 2, part 2, p. 336. See also a paper by A. Ostenfeld, *Beton und Eisen*, vol. 15, 1916.

² Since the diagonals are tension members, their rigidity is small in comparison with that of the struts and can be neglected.

only bending of the vertical is taken into account, then

$$\delta = \frac{Ra^3}{3EI_1},$$

where I_1 is the moment of inertia of one vertical. Taking into account the bending of the floor beam, and using notations indicated in the figure, we obtain

$$\delta = \frac{Ra^3}{3EI_1} + \frac{R(a+b)^2d}{2EI_2},$$

where I_2 is the moment of inertia of the cross section of the floor beam. The force necessary to produce the deflection δ equal to unity is then

$$R_0 = \frac{1}{\frac{a^3}{3EI_1} + \frac{(a+b)^2d}{2EI_2}}$$

and the modulus of the equivalent elastic foundation is

$$\beta = \frac{R_0}{c}$$

where c is the distance between verticals.

In this manner the problem of the stability of the compressed chord of the bridge is reduced to one of buckling of a bar with hinged ends, supported laterally by a continuous elastic medium and axially loaded by a continuous load, the intensity of which is proportional to the distance from the middle.¹ The problem can be solved by taking the differential equation of the deflection curve of the buckled bar and integrating it by the use of infinite series as was explained in the previous article. The same result can be obtained more easily by using the energy method. The deflection curve of the buckled bar in the case of hinged ends can be represented by the series

$$y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + a_3 \sin \frac{3\pi x}{l} + \dots \quad (a)$$

Assuming that the cross section of the bar is constant along its length, and denoting by β the modulus of elastic foundation, which is also considered as constant, the strain energy of bending of the bar,² together with the

¹ In this form the problem of the stability of low-truss bridges was first discussed by F. S. Jasinsky, *loc. cit.*, p. 100. See also French translation. Some corrections of Jasinsky's results have been discussed by the writer by using the energy method, *loc. cit.*, p. 82.

² In applying the energy method, more accuracy in the approximate solution can be obtained by using for the strain energy of bending the expression $\int M^2 dx/2EI$ instead of $(EI/2) \int (d^2y/dx^2)^2 dx$ (see p. 81). This is especially true in the case when only the first approximation is calculated. In the present problem, however, several consecutive approximations will be calculated. The simple expression (b) for strain energy possesses certain advantages in making these calculations.

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strain energy of the foundation, is (see Art. 20)

$$\Delta V = \frac{\pi^4 EI}{4l^3} \sum_{n=1}^{\infty} n^4 a_n^2 + \frac{\beta l}{4} \sum_{n=1}^{\infty} a_n^2. \quad (b)$$

In calculating the work produced by the distributed compressive load during bending, we note that the intensity of this load at any cross section, distance x from the left support (Fig. 78c), is

$$q = q_0 \left(1 - \frac{2x}{l} \right), \quad (c)$$

where q_0 is the intensity of load at the ends. For a truss with parallel chords and a large number of panels, it can be concluded from elementary statics that we can assume for the maximum intensity of the axial load (Fig. 78c)

$$q_0 = \frac{Q}{2h},$$

where Q is the total load on one truss and h the depth of the truss. Considering an element of the upper chord between the two consecutive cross sections mn and m_1n_1 , the axial load to the right of the cross section mn will be displaced toward the immovable support A , owing to the small inclination of this element during buckling, by the amount $\frac{1}{2}(dy/dx)^2 dx$ and will produce the work

$$-\frac{1}{2} \left(\frac{dy}{dx} \right)^2 dx \int_x^l q_0 \left(1 - \frac{2x}{l} \right) dx = \frac{q_0}{2l} x(l-x) \left(\frac{dy}{dx} \right)^2 dx.$$

The total work produced by the compressive load during bending is

$$\Delta T = \frac{q_0}{2l} \int_0^l x(l-x) \left(\frac{dy}{dx} \right)^2 dx.$$

Substituting in this expression the series (a) for y and using the formulas:

$$\begin{aligned} \int_0^l x \cos^2 \frac{m\pi x}{l} dx &= \frac{l^2}{4}; & \int_0^l x^2 \cos^2 \frac{m\pi x}{l} dx &= \frac{l^3}{6} + \frac{l^3}{4m^2\pi^2}; \\ \int_0^l x \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} dx &= 0, & \text{when } m+n \text{ is an even number;} \\ \int_0^l x \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} dx &= -\frac{2l^2}{\pi^2} \frac{m^2 + n^2}{(m^2 - n^2)^2}, & \text{when } m+n \text{ is an odd number;} \\ \int_0^l x^2 \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} dx &= \frac{2l^3}{\pi^2} \frac{m^2 + n^2}{(m^2 - n^2)^2} (-1)^{m+n}; \end{aligned}$$

we finally obtain

$$\Delta T = \frac{q_0}{2} \left[\sum_{n=1}^{\infty} a_n^2 \left(\frac{n^2\pi^2}{12} - \frac{1}{4} \right) - 4 \sum_n \sum_m a_n a_m \frac{nm(m^2 + n^2)}{(m^2 - n^2)^2} \right], \quad (d)$$

where the double series in the parenthesis contains only terms in which the sum $(m+n)$ is even and m is not equal to n . Substituting (b) and (d) in

Eq. (64), we obtain for the maximum compressive force at the middle the following expression:

$$\frac{q_0 l}{4} = \frac{\frac{\pi^4 EI}{8l^2} \sum_{n=1}^{\infty} n^4 a_n^2 + \frac{\beta l^2}{8} \sum_{n=1}^{\infty} a_n^2}{\sum_{n=1}^{\infty} a_n^2 \left(\frac{n^2 \pi^2}{12} - \frac{1}{4} \right) - 4 \sum_n \sum_m a_n a_m \frac{nm(m^2 + n^2)}{(m^2 - n^2)^2}} \quad (e)$$

Now, the problem is to find such relations between the coefficients a_1, a_2, a_3, \dots as to make expression (e) a minimum. Proceeding as in the previous article and setting equal to zero the derivatives of this expression with respect to a_1, a_2, \dots , we finally arrive at a system of homogeneous linear equations in a_1, a_2, \dots of the following type:

$$\left[(n^4 + \gamma)\pi^2 - 2\alpha \left(\frac{n^2 \pi^2}{3} - 1 \right) \right] a_n + 16\alpha \sum_m a_m \frac{nm(m^2 + n^2)}{(m^2 - n^2)^2} = 0, \quad (f)$$

in which, for simplification, the following notations are used:

$$\alpha = \frac{q_0 l}{4} \div \frac{\pi^2 EI}{l^2}, \quad \gamma = \frac{\beta l^4}{\pi^4 EI}. \quad (g)$$

The summation in the second term of Eq. (f) is extended over all values of m different from n such that $(m + n)$ is an even number. Thus, Eq. (f) can be subdivided into two groups, one containing the coefficients a_m with all values of m taken odd and the second with all values of m taken even.

The equations of the first group are:

$$\begin{aligned} \left[(1 + \gamma)\pi^2 - 2\alpha \left(\frac{\pi^2}{3} - 1 \right) \right] a_1 + \alpha \left(\frac{15}{2} a_3 + \frac{65}{18} a_5 + \frac{175}{72} a_7 + \dots \right) &= 0; \\ \frac{15}{2} \alpha a_1 + \left[(3^4 + \gamma)\pi^2 - 2\alpha (3\pi^2 - 1) \right] a_3 + \alpha \left(\frac{255}{8} a_5 + \frac{609}{50} a_7 + \dots \right) &= 0; \\ \frac{65}{18} \alpha a_1 + \frac{255}{8} \alpha a_3 + \left[(5^4 + \gamma)\pi^2 - 2\alpha \left(\frac{25}{3} \pi^2 - 1 \right) \right] a_5 + \alpha \left(\frac{1,295}{18} a_7 + \dots \right) &= 0; \\ \frac{175}{72} \alpha a_1 + \frac{609}{50} \alpha a_3 + \frac{1,295}{18} \alpha a_5 + \left[(7^4 + \gamma)\pi^2 - 2\alpha \left(\frac{49}{3} \pi^2 - 1 \right) \right] a_7 + \dots &= 0; \\ \dots \dots \dots & \quad (h) \end{aligned}$$

The equations of the second group are:

$$\begin{aligned} \left[(2^4 + \gamma)\pi^2 - 2\alpha \left(\frac{4}{3} \pi^2 - 1 \right) \right] a_2 + \alpha \left(\frac{160}{9} a_4 + \frac{15}{2} a_6 + \dots \right) &= 0; \\ \frac{160}{9} \alpha a_2 + \left[(4^4 + \gamma)\pi^2 - 2\alpha \left(\frac{16}{3} \pi^2 - 1 \right) \right] a_4 + \alpha \left(\frac{1,248}{25} a_6 + \dots \right) &= 0; \quad (i) \\ \frac{15}{2} \alpha a_2 + \frac{1,248}{25} \alpha a_4 + \left[(6^4 + \gamma)\pi^2 - 2\alpha \left(\frac{36}{3} \pi^2 - 1 \right) \right] a_6 + \dots &= 0; \\ \dots \dots \dots & \end{aligned}$$

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Buckling of the chord becomes possible when one of the above two systems of equations gives for coefficients a_m a solution different from zero, i.e., when the determinant of system (h) or of system (i) becomes equal to zero. The system (h) corresponds to a symmetrical shape of the buckled bar. The system (i) corresponds to an unsymmetrical shape of the buckled bar.

Let us begin with the case where the rigidity of the elastic medium is very small. In this case the deflection curve of the buckled bar has only one half-wave (see Art. 20) and is symmetrical with respect to the middle. Equations (h) should be used. The first approximation is obtained by taking only the first term in the series (a) and putting $a_3 = a_5 = \dots = 0$. Then, the first equation of (h) will give for a_1 a solution different from zero only if

$$(1 + \gamma)\pi^2 - 2\alpha\left(\frac{\pi^2}{3} - 1\right) = 0,$$

from which

$$\alpha = \frac{\pi^2(1 + \gamma)}{2(\frac{1}{3}\pi^2 - 1)}.$$

Using notations (g), we finally obtain

$$\left(\frac{q_0 l}{4}\right)_{cr} = \frac{\pi^2 EI}{l^2} \frac{\pi^2(1 + \gamma)}{2(\frac{1}{3}\pi^2 - 1)}. \quad (j)$$

If there is no lateral elastic resistance and if the bar is compressed by axial load distributed as shown in Fig. 78c, the quantity γ in Eq. (j) becomes zero [see notations (g)] and we obtain

$$\left(\frac{q_0 l}{4}\right)_{cr} = 2.15 \frac{\pi^2 EI}{l^2}. \quad (k)$$

Thus the critical load is more than twice as large as in the case where the bar is compressed by the loads applied at the ends.

To get a better approximation for the critical compressive force, we take the two terms in expression (a), with coefficients a_1 and a_3 . The corresponding two equations, from system (h), are

$$\begin{aligned} \left[(1 + \gamma)\pi^2 - 2\alpha\left(\frac{\pi^2}{3} - 1\right)\right]a_1 + \frac{15}{2}\alpha a_3 &= 0; \\ \frac{15}{2}\alpha a_1 + [(3 + \gamma)\pi^2 - 2\alpha(3\pi^2 - 1)]a_3 &= 0. \end{aligned}$$

Taking γ equal to zero and equating to zero the determinant of the above two equations, we obtain

$$\left[\pi^2 - 2\alpha\left(\frac{\pi^2}{3} - 1\right)\right]\left[81\pi^2 - 2\alpha(3\pi^2 - 1)\right] - \left(\frac{15}{2}\right)^2 \alpha^2 = 0.$$

Solving this equation for α , we obtain

$$\alpha = 2.06; \quad \left(\frac{q_0 l}{4}\right)_{cr} = 2.06 \frac{\pi^2 EI}{l^2}. \quad (l)$$

By using three terms of the series (a) with the coefficients a_1 , a_3 and a_5 and the three equations of system (h), a third approximation can be calculated. Such calculations show that the error of the second approximation,

given by Eq. (l), is less than 1 per cent so that further approximations are of no practical importance and we can put

$$\left(\frac{q_0 l}{4}\right)_{cr} = 2.06 \frac{\pi^2 EI}{l^2} = \frac{\pi^2 EI}{(0.696l)^2}$$

Thus the reduced length in this case is

$$L = 0.696l.$$

In a similar manner the critical load may be calculated for a chord with small lateral elastic reactions ($\gamma < 3$). Where a greater restraint is supplied by the vertical members of the truss, the buckled form of the chord may have two half-waves and we obtain an inflection point at the middle of the bar. To calculate the critical load in such a case, the system (i) should be used. With a further increase of γ , the buckled bar has three half-waves, and we must again turn to the system of equations (h) in calculating the critical value of the compressive load. In all these cases the critical load can be represented by the equation

$$\left(\frac{q_0 l}{4}\right)_{cr} = \frac{\pi^2 EI}{L^2} \quad (86)$$

in which the reduced length L depends on the rigidity of the elastic foundation. Several values of the ratio L/l are given in the following table:

TABLE 9.—TABLE FOR CALCULATING THE REDUCED LENGTH L

$Bl^4/16EI$	0	5	10	15	22.8	56.5	100	162.8	200	300	500	1,000
L/l	0.696	0.524	0.443	0.396	0.363	0.324	0.290	0.259	0.246	0.225	0.204	0.174

It is seen from the table that, when the rigidity of the elastic foundation increases, the ratio L/l approaches the values obtained before for a uniformly compressed bar (see table on p. 111).

The method developed above for the case of a bar of uniform cross section supported by an elastic medium of a uniform rigidity along the length of the bar can be extended to include cases of chords of variable cross section and cases where the rigidities of the elastic supports vary along the length.¹

25. Buckling of Bars of Variable Cross Section.—An examination of the bending-moment diagram for a buckled bar indicates that a bar of uniform cross section is not the most economical

¹ Several applications of the energy method in design of through bridges are given in the paper by S. Kasarnowsky and D. Zetterholm, *Der Bauingenieur*, vol. 8, p. 760, 1927. See also papers by A. Hrennikoff and by K. Kratochvil in *Publications of the International Association for Bridge and Structural Engineering*, vol. 3, 1935.